Exploring the Distribution of Zeros of the Prime Zeta Function: New Insights Through 3D Visualization and Statistical Analysis

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Abstract

This study focuses on the prime zeta function, an under-explored function in analytic number theory. Special attention is devoted to its zero-free region and the distribution of the zeros. Our research identified over 10000 zeros within the complex rectangle $(0.1, 1.65) \times (0, 10^4)$. Using statistical analysis, we validated some conjectures regarding zeros' distribution patterns (both within and beyond the critical strip), offering new insights into their underlying structure.

Introduction

The prime zeta function $\zeta_{\mathbb{P}}(s)$ is a lesser-studied counterpart of the famous Riemann zeta function. Despite its relevance, the behavior and distribution of its zeros remain relatively unexplored. Recent advancements in computational methods and visualization tools have opened new avenues for analyzing complex functions like $\zeta_{\mathbb{P}}(s)$.

In this work, we focus on the zero distribution of the prime zeta function (both inside and outside its critical strip), emphasizing its zero-free region and its zeros' statistical properties. Additionally, we employ 3D visualization techniques to uncover novel insights into the function's behavior.

Results and Visualization

Our research identified over 10,000 zeros of the prime zeta function within the complex rectangle $(0.1, 1.65) \times (0, 10^4)$ (see Figure 1). The findings suggest a concentration of zeros within the interval $0.3 < \sigma < 0.7$, with a peak in the neighborhood of the line $\sigma = 0.6$. Beyond the critical strip, the density of zeros decreases significantly. Notably, we have proved that the function is zero-free in the half-plane $\sigma > 1.77954465354699...$, where the constant is determined by the root of the equation $\zeta_{\mathbb{P}}(\sigma) = 2^{1-\sigma}$. However, empirically, by our calculations,

$$\sigma_{\max} = \max_{|t|<2\cdot 10^5} \{\sigma \mid \zeta_{\mathbb{P}}(s) = 0\} = 1.682628788045196\ldots .$$

A significant aspect of our study involves 3D visualizations, providing an intuitive understanding of the prime zeta function's complex surfaces. These models illustrate the intersection of real and imaginary components, highlight zero-plane isolines, and reveal intricate patterns in the function's behavior (see Figures 2, 3, and 4). Such visualization aids in identifying patterns and singularities that are otherwise difficult to observe.

Conclusion and Discussion

The results of this study shed light on the intricate dynamics of the prime zeta function. By combining statistical analysis and advanced 3D visualizations, we have identified patterns in the distribution of zeros and clarified the zero-free regions of $\zeta_{\mathbb{P}}(s)$.

Our findings pave the way for further research, including deeper explorations in the critical strip and developing more sophisticated visualization techniques. This work contributes to the broader understanding of complex analytic functions and underscores the value of computational tools in modern number theory.

Visualizations of the Prime Zeta Function

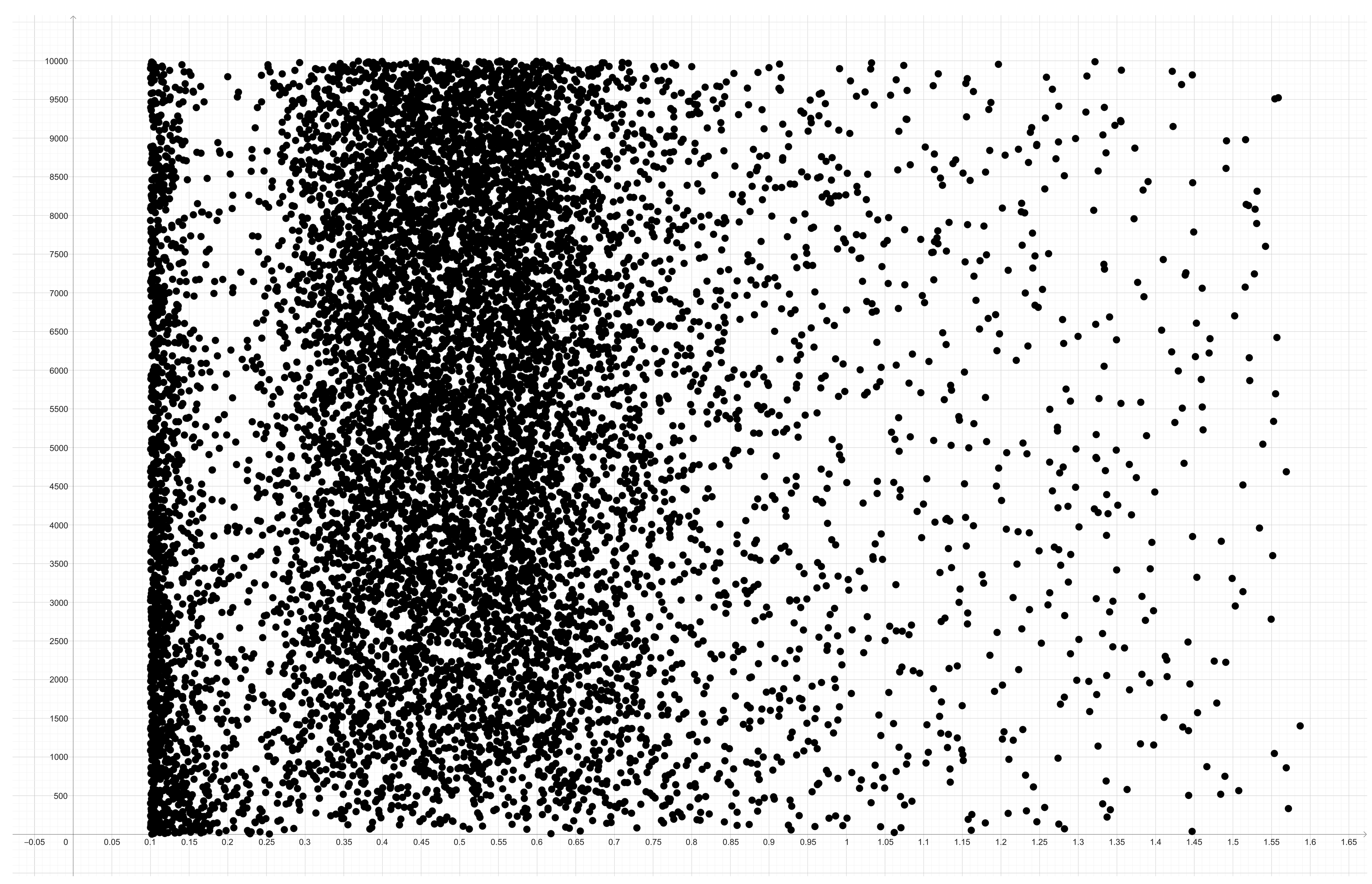

Figure 1: Zeros of $\zeta_{\mathbb{P}}(s)$ in the rectangle $0.1 < \sigma < 1.65$, $0 < t < 10^4$

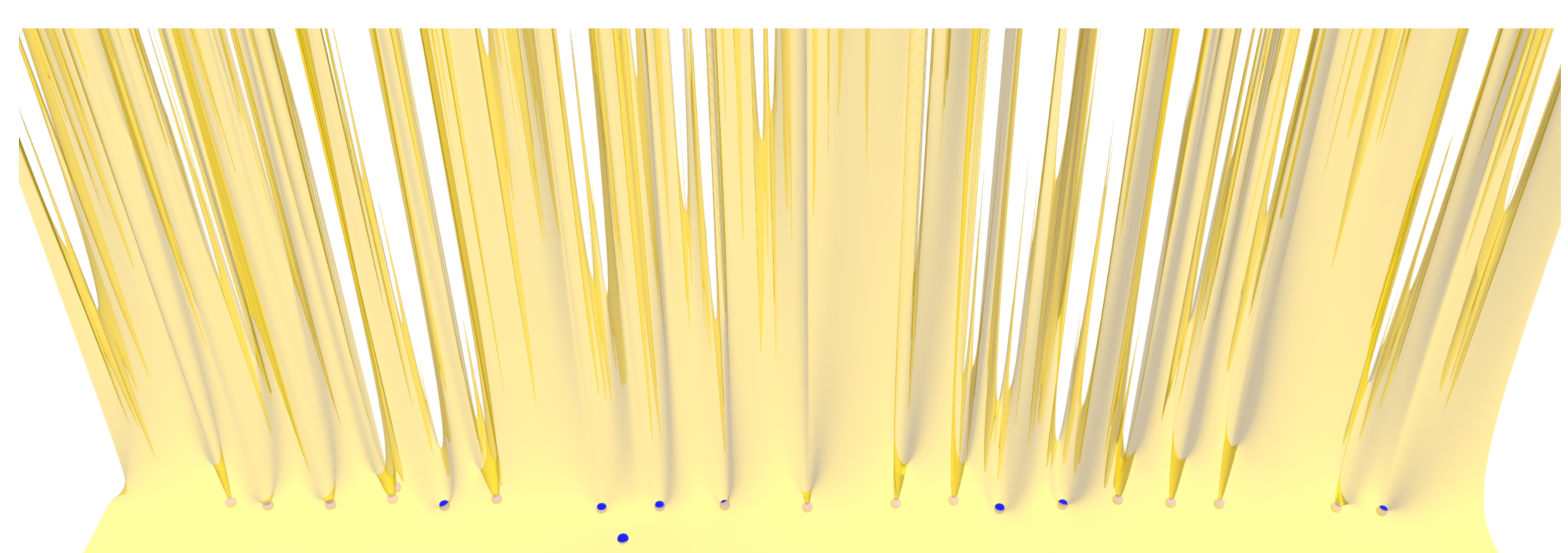

Figure 2: Surface of the target function $|\zeta_{\mathbb{P}}(s)|^2$ in the rectangle $(0.1; 1.4) \times (9980; 10000)$

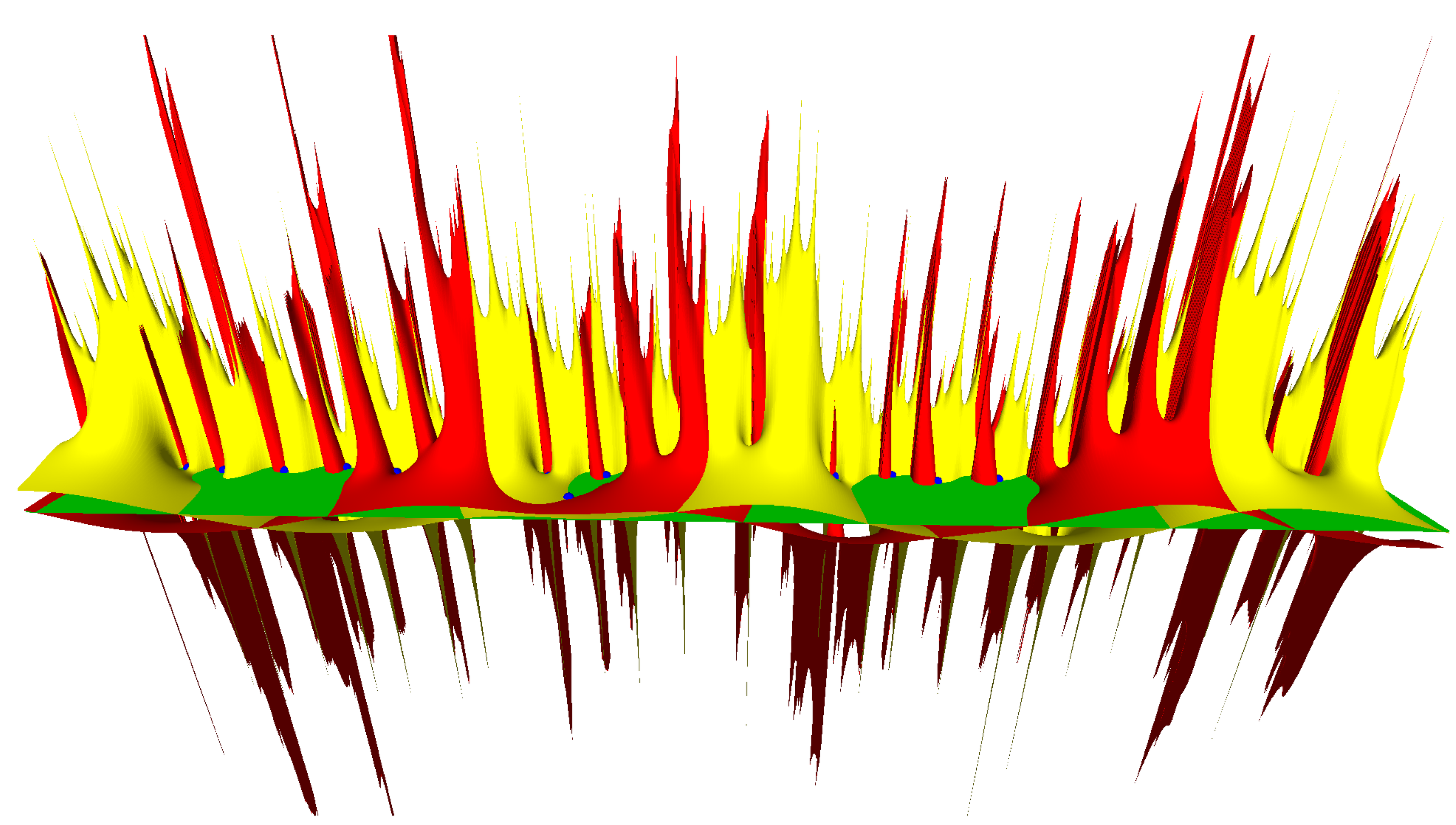

Figure 3: Real and imaginary surfaces of the prime zeta function in the rectangle $(0.1; 1.4) \times (9980; 10000)$

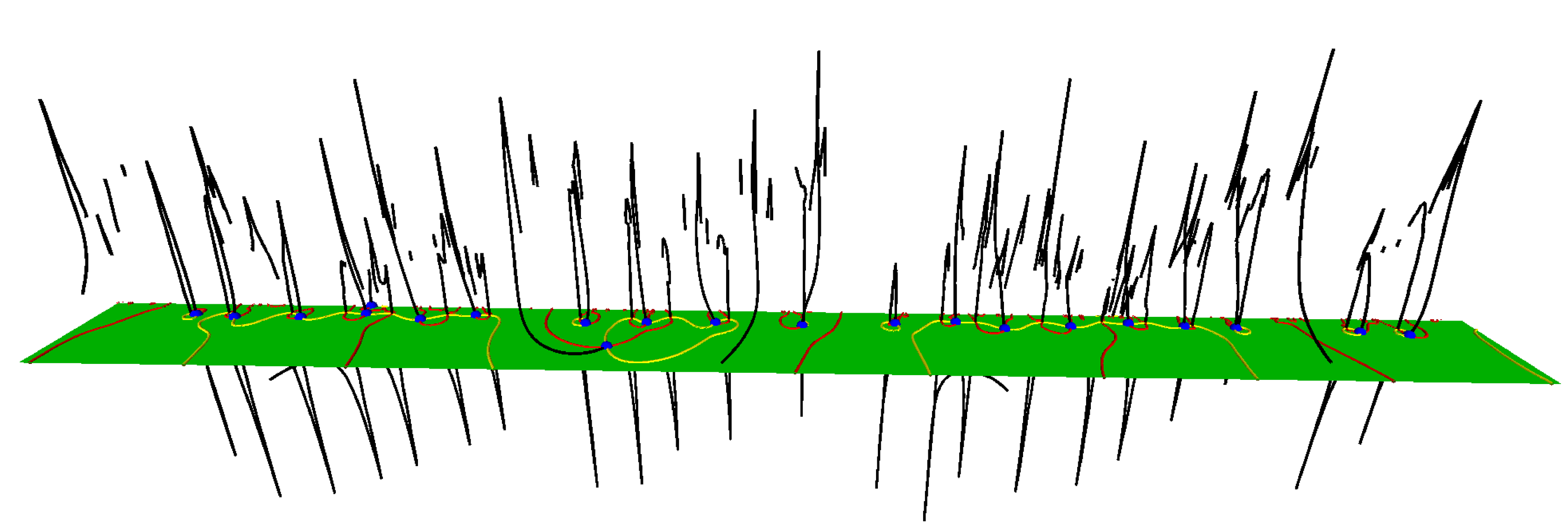

Figure 4: Intersection curves of real and imaginary $\zeta_{\mathbb{P}}(s)$ surfaces in the rectangle $(0.1; 1.4) \times (9980; 10000)$